IV. *On Infinite Series.* *By* Edward Waring, *M. D. F. R. S.* Lucaſian *Profeſſor of Mathematics at* Cambridge.

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1. IN the Paper, which the Royal Society did me the honour to print, on Summation of Series, is given a method of finding the ſum of a ſeries, whoſe general term $\left(\frac{P}{Q}\right)$ (where $\frac{P}{Q}$ is a fraction reduced to its loweſt terms) is a determinate algebraical function of the quantity (z) the diſtance from the firſt term of the ſeries, which always terminates when the ſum of the ſeries can be expreſſed in finite terms.

2. The terms of every infinite ſeries muſt neceſſarily be given by a function of z, or by quantities which can be reduced to a function of z.

3. Let $Q = A \times A' \times A'^{2} \times \,..\, A'^{n} \times B \times B' \times B'^{2} \times \ldots B'^{m} \times C$ $C' \times C'^{2} \times \,..\, C'^{r} \times$ &c. where A', A'^{2}, $A'^{3} \ldots A'^{n}$, are ſucceſſive values of A; that is, reſult from A by writing in it for z reſpectively $z+1, z+2, z+3, \,..\, z+n$; and B', $B'^{\cdot 2}$, $B'^{\cdot 3}$, B'^{m}, reſult from B, by writing in it for z reſpectively $z+1$, $z+2, z+3, \ldots z+m$; but B is not a ſucceſſive value of A; &c. Let the numerator $P = E \,.\, E' \,.\, E'^{2} \,..\, E'^{b-1} \,.\, F \,.\, F' \,.\, F'^{2} \ldots F'^{\cdot k-1} \times L$; E', $E'^{2} \,..\, E'^{b-1}$; F', $F'^{\cdot 2} \,..\, F'^{k-1}$, &c. denoting ſucceſſive values of the quantities E, F, &c. reſpectively; and L, admitting of no diviſor of the formula $K \times K'$, where K' is a ſuc-

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ceſſive value of K; let $L = A \times B \times C \times \&c. \times E^{1b} \times F^{1k} \times \&c. \times p' \times q' \times r' \times \&c. - A^{1n} \times B^{1m} \times C^{1r} \times \&c. \times E \times F \times \&c. \times p \times q \times r \times \&c.$ where p', q', r', &c. are irrational quantities and ſucceſſive values of p, q, r, &c. The factors A B, C, &c. E^{1b}, F^{1k}, &c. being given, the factors p', q', r', &c. into which they are multiplied in the quantity L will eaſily be deduced by deducting the preceding irrational factors contained in A, B, C, &c. E^{1b}, F^{1k}, &c. from the correſpondent irrational factors contained in L; and in the ſame manner, from the factors A^{1n}, B^{1m}, &c. E, F, &c. can be deduced the irrational factors of the preceding p, q, r, &c.

Aſſume for the ſum of the ſeries ſought the quantity

$$\frac{E \times E' \times E^{12} \,..\, E^{1b-1} \times F \times F' \,..\, F^{1k-1} \times \&c.}{A \,.\, A' \,.\, A^{12} \,..\, A^{1n-1} \times B \times B' \times B^{1.2} \,..\, B^{1m-1} \times C \times C' \,..\, C^{1.r-1} \times \&c.} \times p \times q \times r \times s \times \&c. \; (\alpha z^{m'} + \beta z^{m'-1} + \gamma z^{m'-2} + \&c.) = V;$$

where m' is a whole number, and α, β, γ, &c. are co-efficients to be inveſtigated; write in V for z its ſucceſſive value $z+1$, and let the reſult be W; reduce the difference $W - V$ into a fraction in its loweſt terms, and make the co-efficients of the correſpondent terms of the reſulting fraction $= (W - V)$ and of the given fraction equal to each other, if poſſible; and thence may be deduced the ſum of the ſeries required.

4. This ſeries will terminate if the ſum ſought can be expreſſed by a finite determinate function of z; if not, it will proceed *in infinitum*, and may be expreſſed either by a ſeries aſcending or deſcending according to the dimenſions of z.

5. If any factor, A or B, or C, &c. have no ſucceſſive one in the denominator; or if the greateſt dimenſions of z in the denominator be greater than its greateſt dimenſions in the numerator by 1, then the ſum of the ſeries is not a finite algebraical function of z.

6. If in the denominator are deficient ſome intermediate ſucceſſive factors, multiply both the numerator and denominator by thoſe deficient factors, and they are ſupplied: for example, let $A \times A''' \times A''''' \times$ &c. be factors of the denominator, in which are deficient the factors A', A'', A'''', &c. multiply both numerator and denominator by the content $A' \times A'' \times A'''' \times$ &c. and they are reſtored.

7. If in the denominator is contained the content $A^{l} \times A'^{l'} \times A''^{l''} \,..\, A'^{n^{l'n}} = H$, in which λ is the leaſt of the indices l, l', l'', &c.: aſſume $A^{\lambda} \times A'^{\lambda} \times A''^{\lambda} \ldots A^{1n^{\lambda n}} \times \alpha = H$, and in the ſame manner reduce the factors in α; then aſſume for the denominator $A^{\lambda} \times A'^{\lambda} \times A''^{\lambda} \ldots A'^{n-1^{\lambda}} \times$ &c.: for example, let the contents be $A^{3} \times A'^{5} \times A''^{2} \times {}'''^{5} \times$ &c. $= H$, then is 2 the leaſt index, and conſequently the content reduced as before taught will be $A^{2} \,.\, A'^{2} \,.\, A''^{2} \,.\, A'''^{2} \times \alpha = A^{2} \,.\, A'^{2} \,.\, A''^{2} \,.\, A'''^{2} \times A \,.\, A' \,.\, A''' \times A'^{2} \,.\, A'''^{2}$; but between the factors A' and A''' is deficient the factor A''; and between the factors A'^{2} and A'''^{2} is deficient the factor A''^{2}; multiply the numerator and denominator of the given fraction by the deficient factors $A'' \times A''^{2}$; aſſume for the denominator $A^{2} \,.\, A'^{2} \,.\, A''^{2} \times A \,.\, A' \,.\, A'' \times A'^{2} \,.\, A''^{2} \times$ &c. $= A^{3} \,.\, A'^{5} \,.\, A''^{5} \times$ &c. &c.

8. If the greateſt index of the content H is contained in one factor only, then the ſum of the ſeries cannot be expreſſed in finite terms of the quantity z.

9. The ſame may be applied to the contents of the ſeveral ſucceſſive values of the quantities B, C, &c. in the denominator: for example, let the general term be $\frac{1}{1 \,.\, 2 \,.\, 3 \,..\, \overline{z-2} \times z}$; multiply it into $z - 1$ to compleat the deficient term, and it reſults

$\frac{z-1}{1.2.3..z}$; aſſume, by the preceding method for the ſum of the ſeries the quantity $\frac{1}{1.2.3..z-1}$, of which the ſucceſſive term is $\frac{1}{1.2.3..z}$, and their difference $\frac{1}{1.2..z-1} - \frac{1}{1.2..z}$ $= \frac{1}{1.2..z-2 \times z}$ the given term.

2. Let the term be Ne^z and e leſs than 1, which is the term of a geometrical ſeries; then will the ſum of the infinite ſeries be $\frac{N}{1-e} \times e^z$, beginning from the term whoſe diſtance from the firſt is z; for the difference between the two ſucceſſive ſums $=$ $\frac{N}{1-e}(e^z - e^{z+1}) = Ne^z$ the given term.

3. Let the general term be $\frac{(z+n+1-e^n \times \overline{z+1})a}{z+1 . z+n+1} \times e^{z+1}$; aſſume for the ſum of the ſeries the ſubſequent quantity $(z+1 . z+2 . z+3 .. z+n)^{-1} \times e^z \times (\alpha + \beta z + \gamma z^2 .. z^{n-1})$ and by the preceding method the co-efficients α, β, γ, &c. may be found: the ſum is known to be $= \frac{a}{z+1} \times e^{z+1} + \frac{a}{z+2} \times e^{z+2} + \frac{a}{z+3} \times e^{z+3}$ $.. \frac{a}{z+n} e^{z+n}$, which can eaſily be reduced to the preceding formula.

If the general term be $\frac{T'-T}{T \times T'}$ or $\frac{PQ'-QP'}{QQ'}$; where T and T′, P and P′, Q and Q′, are ſucceſſive terms; then will the ſums of the ſerieſes be $\frac{1}{T}$ or $\frac{P}{Q}$ properly corrected.

10. If the function expreſſing the general term contain in the denominator a factor or factors, which have no ſucceſſive one; reduce the factor or factors into an infinite ſeries proceeding according to the dimenſions of z, and thence, by the method before given, find the ſum of the ſeries. The ſame method

may be purſued, when the denominator of a fluxion, which is a function of x multiplied into $\dot{x}$ contains the ſimple power only of a factor or factors; reduce the factor or factors into an infinite ſeries, proceeding according to the dimenſions of x, and by the known methods find the fluent of the fluxion.

11. The fluent of the fluxion or ſum of the ſeries may be deduced alſo from the ſubſequent propoſitions, from which may be inveſtigated many ſerieſes, whoſe ſums are known.

1. Let $p\dot{P}=Q\dot{q}$, $q\dot{Q}=R\dot{r}$, $r\dot{R}=S\dot{s}$, $s\dot{S}=T\dot{t}$, &c.; then $\int P\dot{p}=$ $Pp-Qq+Rr-Ss+Tt-$&c. if only the ſeries converges.

2. Let $pP'+p'P'=Qq'$, $qQ'+q'Q'+=Rr'$, $rR'+r'R'=$ Ss', $sS'+s'S'=Tt'$, &c. where P', p', Q', q', &c. denote the increments of the quantities P, p, Q, q, &c. reſpectively, then will the integral of the increment $(Pp')=Pp-Qq+Rr-Ss+$ &c. if only the ſeries converges.

Ex. 1. $\int x^{-n}\dot{x}=\int\frac{x^m\dot{x}}{x^{n+m}}=\frac{1}{x^{n-1}}\left(\frac{1}{m+1}+\frac{n+m}{r'+1}A+\frac{n+r'}{s'+1}B+\frac{n+s'}{t'+1}\right.$ $\left.C+\frac{n+t'}{T+1}D+\text{&c.}\right)=\frac{1}{1-nx^{n-1}}$; whence $\frac{1}{1-n}=\frac{1}{m+1}+\frac{n+m}{r'+1}\times\frac{1}{m+1}$ $\frac{n+r'}{s'+1}\times\frac{n+m}{r'+1}\times\frac{1}{m+1}+$&c. In this example $\dot{p}=x^m\dot{x}$, $P=x^{-n-m}$, $\dot{q}=x^{r'}\dot{x}$, $Q=x^{-n-r'}$, $\dot{r}=x^{s'}\dot{x}$, &c.

Ex. 2. $\int\frac{x^m\dot{x}}{x^{n+r}}=\int x^{m-n-r}\dot{x}=\frac{1}{m-n-r+1}x^{m-n-r+1}=x^{m-n-r+1}$ $\left(\frac{1}{m+1}+\frac{1}{m+1}\times\frac{n+r}{m+2}+\frac{1}{m+1}\cdot\frac{n+r}{m+2}\cdot\frac{n+r+1}{m+3}+\text{&c.}\right)$; whence $\frac{1}{m-n-r+1}=\frac{1}{m+1}+\frac{n+r}{m+2}A+\frac{n+r+1}{m+3}B+$&c. In both theſe examples the letters A, B, C, &c, denote the preceding terms.

Ex. 3. $\int\frac{x^m\dot{x}}{1+x}=\frac{x^{m+1}}{1+x}\left(\frac{1}{m+1}+\frac{1}{m+1\,.\,m+2}\times\frac{x}{1+x}+\frac{2}{m+1\,.\,m+2\,.\,m+3}\times\right.$ $\left.\frac{x^2}{(1+x)^2}+\frac{2\cdot3}{m+1\,.\,m+2\,.\,m+3\,.\,m+4}\cdot\frac{x^3}{(1+x)^3}+\text{&c.}\right)$

Ex. 4. $\int\frac{a'x^{m}\dot{x}}{(a+bx^{n}+cx^{2n})^{b}}=\frac{1}{m+1}\,\frac{a'x^{m+1}}{(a+bx^{n}+cx^{2n})^{b}}+\frac{a'x^{m+1}}{m+1\,.\,(a+bx^{n}+cx^{2n})^{b+1}}$
$\times\left(\frac{nbbx^{n-1}}{m+n+1}+\frac{2nhcx^{2n-1}}{m+2n+1}\right)+\&c.$

12. Let the general term of an infinite ſeries be

$$\frac{A}{z+\alpha\,.\,z+\alpha+1\,.\,z+\alpha+2\,.\,.\,z+\alpha+n\times z+\beta\,.\,z+\beta+1\,.\,z+\beta+2\,.\,.\,z+\beta+}$$
$$\frac{}{m\times z+\gamma\,.\,z+\gamma+1\,.\,z+\gamma+2\,.\,.\,z+\gamma+r\times z+\delta\,.\,z+\delta+1\,.\,z+\delta+2\,.\,.}$$

$\frac{}{z+\delta+s\times\&c.}$; where $\alpha-\beta$, $\alpha-\gamma$, $\beta-\gamma$, &c. are not whole numbers; the ſum of the ſeries can always be expreſſed in finite terms; if the ſum of the fractions $\frac{1}{1\,.\,2\,.\,3\,.\,.\,n}\times$

$$\frac{1}{\beta-\alpha\,.\,\beta-\alpha+1\,.\,\beta-\alpha+2\,.\,.\,\beta-\alpha+m\times\gamma-\alpha\,.\,\gamma-\alpha+1\,.\,\gamma-\alpha+2\,.\,.\,.\,\gamma-\alpha+r}$$
$$\frac{}{\times\delta-\alpha\,.\,\delta-\alpha+1\,.\,\delta-\alpha+2\,.\,.\,\delta-\alpha+s\times\&c.}-\frac{1}{1\times1\,.\,2\,.\,3\,.\,.\,n-1}\times$$
$$\frac{1}{\beta-\alpha-1\,.\,\beta-\alpha\,.\,\beta-\alpha+1\,.\,\beta-\alpha+2\,.\,.\,.\,\beta-\alpha+m-1\times\gamma-\alpha-1\,.\,\gamma-\alpha\,.\,\gamma-\alpha+1}$$
$$\frac{}{.\,.\,\gamma-\alpha+r-1\times\delta-\alpha-1\,.\,\delta-\alpha\,.\,\delta-\alpha+1\,.\,.\,\delta-\alpha+s-1\times\&c.}+$$
$$\frac{1}{1\,.\,2\times1\,.\,2\,.\,3\,.\,.\,n-2}\times\frac{1}{\beta-\alpha-2\,.\,\beta-\alpha-1\,.\,\beta-\alpha\,.\,\beta-\alpha+1\,.\,.\,\beta-\alpha+n-2\times\gamma-\alpha}$$
$$\frac{}{-2\,.\,\gamma-\alpha-1\,.\,\gamma-\alpha\,.\,.\,.\,\gamma-\alpha+n-2\times\delta-\alpha-2\,.\,\delta-\alpha-1\,.\,\delta-\alpha\,.\,.\,\delta-\alpha+n-2\times\&c.}$$
$$-\frac{1}{1\,.\,2\,.\,3\times1\,.\,2\,.\,.\,n-3}\times\frac{1}{\beta-\alpha-3\,.\,\beta-\alpha-2\,.\,.\,\beta-\alpha+n-3\times\gamma-\alpha-3\,.\,\gamma-}$$
$$\frac{}{\alpha-2\,.\,.\,\gamma-\alpha+n-3\times\delta-\alpha-3\,.\,\delta-\alpha-2\,.\,.\,\delta-\alpha+s-3\times\&c.}+\ldots\frac{1}{1\,.\,2\,.\,3\,.\,.\,n}\times$$
$$\frac{1}{\beta-\alpha-n\,.\,\beta-\alpha-n+1\,.\,.\,\beta-\alpha-n+m\times\gamma-\alpha-n\,.\,\gamma-\alpha-n+1\,.\,.\,\gamma-\alpha-n+r\times}$$
$$\frac{}{\delta-\alpha-n\,.\,\delta-\alpha-n+1\,.\,.\,\delta-\alpha-n+s\times\&c.}=0;\ \frac{1}{1\,.\,2\,.\,3\,.\,.\,m}\times\frac{1}{\alpha-\beta\,.\,\alpha-\beta+}$$
$$\frac{}{1\,.\,\alpha-\beta+2\,.\,.\,\alpha-\beta+m\times\gamma-\beta\,.\,\gamma-\beta+1\,.\,.\,\gamma-\beta+r\times\delta-\beta\,.\,\delta-\beta+1}$$
$$\frac{}{.\,\delta-\beta+2\,.\,.\,\delta-\beta+s\times\&c.}-\frac{1}{1\times1\,.\,2\,.\,3\,.\,.\,m-1}\times\frac{1}{\alpha-\beta-1\,.\,\alpha-\beta\,.\,.\,\alpha-\beta+}$$

$n-1$

$$\frac{1}{n-1\times\gamma-\beta-1\,.\,\gamma-\beta\,.\,.\,\gamma-\beta+r-1\times\delta-\beta-1\,.\,\delta-\beta\,.\,.\,\delta-\beta+s-1\times\&c.}+$$

$$\frac{1}{1\,.\,2\times1\,.\,2\,.\,3\,.\,.\,m-2}\times\frac{1}{\alpha-\beta-2\,.\,\alpha-\beta-1\,.\,\alpha-\beta\,.\,.\,\alpha-\beta+n-2\times\gamma-\beta-2}$$

$$\frac{1}{.\,\gamma-\beta-1\,.\,.\,\gamma-\beta+r-2\times\delta-\beta-2\,.\,\delta-\beta-1\,.\,.\,\delta-\beta+s-2\times\&c.}-$$

$$\frac{1}{1\,.\,2\,.\,3\times1\,.\,2\,.\,3\,.\,.\,m-3}\times\frac{1}{\alpha-\beta-3\times\alpha-\beta-2\times\alpha-\beta-1\,.\,.\,\alpha-\beta+n-3\times\gamma-}$$

$$\frac{1}{\beta-3\,.\,\gamma-\beta-2\,.\,\gamma-\beta-1\,.\,.\,\gamma-\beta+r-3\times\delta-\beta-3\,.\,.\,\delta-\beta+s-3\times\&c.}\;.\;.\;.$$

$$\frac{1}{1\,.\,2\,.\,3\,.\,4\,.\,.\,m}\times\frac{1}{\alpha-\beta-m\,.\,\alpha-\beta-m+1\,.\,.\,\alpha-\beta+n-m\times\gamma-\beta-m\,.\,\gamma-\beta-}$$

$$\frac{1}{m+1\,.\,.\,\gamma-\beta-m+r\times\delta-\beta-m\,.\,\delta-\beta-m+1\,.\,.\,\delta-\beta-m+s\times\&c.}=0;$$

$$\frac{1}{1\,.\,2\,.\,3\,.\,.\,r}\times\frac{1}{\alpha-\gamma\,.\,\alpha-\gamma+1\,.\,.\,\alpha-\gamma+n\times\beta-\gamma\,.\,\beta-\gamma+1\,.\,\beta-\gamma+m}$$

$\overline{\times\delta-\gamma\,.\,\delta-\gamma+1\,.\,.\,\delta-\gamma+s\times\&c.}$ - &c. $=0$, &c. $=0$; or, to explain it otherwiſe, aſſume the quantities α, $\alpha+1$, $\alpha+2$, $\alpha+3$, . . . $\alpha+n$; β, $\beta+1$, $\beta+2$, . . $\beta+m$; γ, $\gamma+1$, $\gamma+2$, . . $\gamma+r$; δ, $\delta+1$, $\delta+2$, . . $\delta+s$; &c. ſubtract one α from all the remaining quantities $\alpha+1$, $\alpha+2$, . $\alpha+n$; β, $\beta+1$, . . $\beta+m$; γ, $\gamma+1$, . . $\gamma+r$; δ, $\delta+1$, . . $\delta+s$; &c. and multiply all the differences reſulting 1, 2, 3, . . n; $\beta-\alpha$, $\beta-\alpha+1$, . . $\beta-\alpha+m$; $\gamma-\alpha$, $\gamma-\alpha+1$, . . $\gamma-\alpha+2$, . . $\gamma-\alpha+r$; $\delta-\alpha$, $\delta-\alpha+1$, . . $\delta-\alpha+s$, &c. into each other, and call the content p^{α}. In the ſame manner ſubtract $\alpha+1$ from all the remaining quantities α, $\alpha+2$, $\alpha+3$, . . $\alpha+n$; β, $\beta+1$, . . $\beta+m$; γ, $\gamma+1$, . . $\gamma+r$; δ, $\delta+1$, . . $\delta+s$; &c.; and let the remainders -1, 1, 2, 3, . . $n-1$; $\beta-\alpha-1$, $\beta-\alpha$, $\beta-\alpha+1$, . . $\beta-\alpha+m-1$; $\gamma-\alpha-1$, $\gamma-\alpha$, $\gamma-\alpha+1$, . . $\gamma-\alpha+r-1$; $\delta-\alpha-1$, $\delta-\alpha$, $\delta-\alpha+1$, . . $\delta-\alpha+s-1$, &c. be multiplied together, and their content be called $p^{\alpha+1}$. In the ſame manner ſubtract $\alpha+2$, $\alpha+3$, . . $\alpha+n$ reſpectively from all the remaining quantities, and let the differences reſulting be multiplied together, and their reſpective contents be called $p^{\alpha+2}, p^{\alpha+3}, p^{\alpha+4}, \ldots p^{\alpha+n}$; then, if the ſum of the ſeries can be found, will $\frac{1}{p^{\alpha}} + \frac{1}{p^{\alpha+1}} + \frac{1}{p^{\alpha+2}} + \frac{1}{p^{\alpha+3}} \ldots \frac{1}{p^{\alpha+n}} = 0$.

In the ſame manner ſubtract $\beta, \beta+1, \beta+2, \ldots \beta+m$ reſpectively from all the remaining quantities, and multiply their reſpective remainders into each other, and call their contents reſpectively $p^{\beta}, p^{\beta+1}, p^{\beta+2}, \ldots p^{\beta+m}$, then will $\frac{1}{p^{\beta}} + \frac{1}{p^{\beta+1}} + \frac{1}{p^{\beta+2}} \ldots \frac{1}{p^{\beta+m}} = 0$.

Subtract $\gamma, \gamma+1, \gamma+2, \ldots \gamma+r$ reſpectively from all the remaining quantities, and multiply their reſpective remainders into each other, and call their contents reſpectively $p^{\gamma}, p^{\gamma+1}, p^{\gamma+2}, \ldots p^{\gamma+r}$; then will $\frac{1}{p^{\gamma}} + \frac{1}{p^{\gamma+1}} + \frac{1}{p^{\gamma+2}} \ldots \frac{1}{p^{\gamma+r}} = 0$.

Subtract $\delta, \delta+1, \delta+2, \ldots \delta+s$ from all the remaining quantities, and multiply their reſpective remainders into each other, and call their contents reſpectively $p^{\delta}, p^{\delta+1}, p^{\delta+2}, p^{\delta+s}$; then will $\frac{1}{p^{\delta}} + \frac{1}{p^{\delta+1}} + \frac{1}{p^{\delta+2}} + \frac{1}{p^{\delta+s}} = 0$; and ſo on; and, *vice verſâ*, if the ſum of the above-mentioned fractions be reſpectively $= 0$; then the ſum of the ſeries, whoſe general term is the given one, can be found; otherwiſe not.

13. If the ſum of the ſeries, whoſe general term is

$$\frac{1}{z+\alpha \,.\, z+\alpha+1 \ldots z+\alpha+n \times z+\beta \,.\, z+\beta+1 \ldots z+\beta+m \times z+\gamma \,.\, z+\gamma+1 \ldots z+\gamma+r \times z+\delta \,.\, z+\delta+1 \ldots z+\delta+s \times \&c. = \mathrm{H}},$$

can be found in finite terms; then the ſum of a ſeries, whoſe general term is $\frac{az^{l} + bz^{l-1} + cz^{l-2} + \&c.}{\mathrm{H}}$ (where l denotes an affirmative number) can be alſo expreſſed in finite terms.

14. Let $A\times\overline{z+\beta}\,.\,\overline{z+\gamma}\,.\,\overline{z+\delta}\times\overline{z+\varepsilon}\times\&c.+B\times\overline{z+\alpha}\times\overline{z+\gamma}\,.\,\overline{z+\delta}\,.\,\overline{z+\varepsilon}\,.\,\&c.+C\times\overline{z+\alpha}\,.\,\overline{z+\beta}\times\overline{z+\delta}\,.\,\overline{z+\varepsilon}\,.\,\&c.+D\times\overline{z+\alpha}\,.\,\overline{z+\beta}\,.\,\overline{z+\gamma}\times\overline{z+\varepsilon}\,.\,\&c.+E\times\overline{z+\alpha}\,.\,\overline{z+\beta}\,.\,\overline{z+\gamma}\,.\,\overline{z+\delta}\times\&c.=1$, whatever may be the value of z; then will $A=\frac{1}{\beta-\alpha\,.\,\gamma-\alpha\,.\,\delta-\alpha\,.\,\&c.}$, $B=\frac{1}{\alpha-\beta\,.\,\gamma-\beta\,.\,\delta-\beta\,.\,\&c.}$, $C=\frac{1}{\alpha-\gamma\,.\,\beta-\gamma\,.\,\delta-\gamma\,.\,\&c.}$, $D=\frac{1}{\alpha-\delta\,.\,\beta-\delta\,.\,\gamma-\delta\,.\,\&c.}$, &c.

15. Let the general term of a ſeries be $\frac{1}{H}\times e^{z}$, where e is leſs than 1; the ſum of the ſeries can be expreſſed in finite terms, when the above-mentioned quantities $\frac{1}{p\alpha}+\frac{\frac{1}{e}}{p\alpha+1}+\frac{\frac{1}{e^{2}}}{p\alpha+2}+\frac{\frac{1}{e^{3}}}{p\alpha+3}\,.\,.+\frac{\frac{1}{e^{n}}}{p\alpha+n}=0$, $\frac{1}{p\beta}+\frac{\frac{1}{e}}{p\beta+1}+\frac{\frac{1}{e^{2}}}{p\beta+2}+\frac{\frac{1}{e^{3}}}{p\beta+3}\ldots+\frac{\frac{1}{e^{m}}}{p\beta+m}=0$, $\frac{1}{p\gamma}+\frac{\frac{1}{e}}{p\gamma+1}+\frac{\frac{1}{e^{2}}}{p\gamma+2}\;\ldots\;\frac{\frac{1}{e^{r}}}{p\gamma+r}=0$, $\frac{1}{p\delta}+\frac{\frac{1}{e}}{p\delta+1}+\frac{\frac{1}{e^{2}}}{p\delta+2}+\frac{\frac{1}{e^{3}}}{p\delta+3}\ldots\frac{\frac{1}{e^{s}}}{p\delta+s}=0$, &c. $=0$. This never happens unleſs $e=1$.

16. If the general term be any rational function of z into the exponential e^{z}, viz.

$$\frac{az^{l}+bz^{l-1}+cz^{l-2}+\&c.}{\overline{z+\alpha}^{m}\times\overline{z+\alpha+b}^{m'}\times\overline{z+\alpha+k}^{m''}\times\&c.\times\overline{z+\beta}^{n}\times\overline{z+\beta+b'}^{n'}\times\overline{z+\beta+k'}^{n''}\times\&c.\times\overline{z+\gamma}^{r}\times\overline{z+\gamma+b''}^{r'}\times\overline{z+\gamma+k''}^{r''}\times\&c.}\times e^{z}=K,$$

where b, b', b'', &c. k, k', k'', &c. l, m, m', m'', &c. n, n', n'', &c. r, r', r'', &c. &c. denote whole numbers, and neither

$\alpha-\beta$, nor $\alpha-\gamma$, nor $\beta-\gamma$, &c. are whole numbers: let m be the greateſt of the indices m, m', m'', &c.; n the greateſt of the indices n, n', n'', &c.; r the greateſt of the indices r, r', r'', &c.; then, from the ſums of the ſerieſes given, whoſe general terms are $\frac{e^z}{(z+\alpha)^m}$, $\frac{e^z}{(z+\alpha)^{m-1}}$, $\frac{e^z}{(z+\alpha)^{m-2}}$, . . . $\frac{e^z}{(z+\alpha)^2}$, $\frac{e^z}{z+\alpha}$; $\frac{e^z}{(z+\beta)^n}$, $\frac{e^z}{(z+\beta)^{n-1}}$ . . $\frac{e^z}{z+\beta}$; $\frac{e^z}{(z+\gamma)^r}$, $\frac{e^z}{(z+\gamma)^{r-1}}$, . . . $\frac{e^z}{z+\gamma}$, &c. can be deduced the ſum of the above ſeries, whoſe general term is given above; multiply each of theſe terms into unknown co-efficients e', f, g, h, &c.; then reduce them to a common denominator, which is the ſame as the denominator of the given general term, and add them together, and make the correſpondent terms of the ſum reſulting equal to the correſpondent terms of the numerator $az^l+bz^{l-1}+cz^{l-2}+$&c. of the given general term; and from the equations reſulting can be deduced the co-efficients e', f, g, h, &c. and thence from the given ſums the ſum of the ſeries required.

Approximations to the ſums of the ſerieſes may be deduced from the methods given in the *Meditationes Analyticæ.* The ſum of ſome few caſes have been given from the periphery of the circle: for example, when α and m are whole numbers, and $e=1$; or, more particularly, when $m=2$ and $e=\frac{1}{2}$, and ſome other particular caſes, which may be with nearly the ſame facility calculated from approximations; the caſes given indeed are ſo few, unleſs when $e=1$, that they can very rarely be applied.

17. If the dimenſions of z in the numerator be equal or greater than its dimenſions in the denominator; that is, l be equal or greater than $m+m'+m''+$&c. $+n+n'+n''+$&c. $+r+r'+r''$

+

$+$ &c. &c. reduce the fractions to a mixed number, ſo that the dimenſions of z in the numerator of the fraction be leſs than its dimenſions in the denominator, and the integral part be $Ae^{z}+Bze^{z}+Cz^{2}e^{z}+Dz^{3}e^{z}+\ldots Hz^{1m}e^{z}$: the ſum of the infinite ſeries whoſe term is $Ae^{z}+Bze^{z}+Cz^{2}e^{z}+Dz^{3}e^{z}+Ez^{4}e^{z}+\ldots+Hz^{m}e^{z}=\frac{Ae}{1-e}+B\left(\frac{1}{(1-e)^{2}}-\frac{1}{1-e}\right)+Ce\left(\frac{1.2}{(1-e)^{3}}-\frac{1}{(1-e)^{2}}\right)+De\left(\frac{1.2.3}{(1-e)^{4}}-\frac{\overline{1+2}\times 1.2}{(1-e)^{3}}+\frac{1}{(1-e)^{2}}\right)+Ee\left(\frac{1.2.3.4}{(1-e)^{5}}-\&c.\right)$: the ſum of a ſeries (whoſe general term is $z^{m}e^{z}$) $=\frac{1.2.3.4..m}{(1-e)^{m+1}}e-\frac{{}^{1}S^{m-1}}{(1-e)^{m}}1.2.3..\overline{m-1}e+\frac{{}^{1}S^{m-1}\times{}^{1}S^{m-2}-{}^{2}S^{m-1}}{(1-e)^{m-1}}\times 1.2.3.\overline{m-2}.e\ldots+\frac{L}{(1-e)^{m-b}}\times 1.2.3..\overline{m-b-1}\ldots\pm\frac{1}{(1-e)^{2}}$, where L is equal to the ſum of all quantities of the following ſort, ${}^{\alpha}S^{m-1}\times{}^{\beta}S^{m-\alpha-1}\times{}^{\gamma}S^{m-\alpha-\beta-1}\times{}^{\delta}S^{m-\alpha-\beta-\gamma-1}\times{}^{\epsilon}S^{m-\alpha-\beta-\gamma-\delta-1}\times$ &c. where α, β, γ, δ, &c. are whole affirmative numbers; (in the preceding notation by ${}^{\pi}S^{\varrho}$ is deſigned the ſum of the contents of every π of the following numbers $1, 2, 3, 4, 5, \ldots \varrho$); and $\alpha+\beta+\gamma+\delta+\epsilon+\&c.=b+1$; the above-mentioned product ${}^{\alpha}S^{m-1}\times{}^{\beta}S^{m-\alpha-1}\times$ &c. is to be taken affirmative or negative, according as the number of letters α, β, γ, δ, &c. is even or uneven.

The ſum of the ſeries $z^{m}\times e^{z}$ may alſo be found by aſſuming for it $(az^{m}+bz^{m-1}+cz^{m-2}+dz^{m-3}\ldots k)e^{z}$; then, finding its ſucceſſive term $(a\times\overline{z+1}^{m}e+b\overline{z+1}^{m-1}e+c\overline{z+1}^{m-2}e+\&c.)\,e^{z}$, and taking the difference between it and the aſſumed quantity, there reſults $(a\times\overline{e-1}\,z^{m}+\overline{mae+be-1}\,z^{m-1}+\&c.)\,e^{z}$; by equating it to the given term $z^{m}e^{z}$ are deduced the ſubſequent equations $a\times\overline{e-1}=1$, $mae+be-1=0$, &c. whence

$a=\frac{1}{e-1}$, $b=\frac{1}{e}-ma$, &c.

If $e=1$, then aſſume $az^{m+1}+bz^{m}+$&c. for the ſum ſought, which rule was firſt taught by M. J. BERNOULLI.

If the term be $z^{m}f^{bz}$; for f^{b} ſubſtitute e, and there reſults $z^{m}e^{z}$ the ſame as before.

18. Let $\mathrm{P}=\mathrm{A}+\mathrm{B}x^{n}+\mathrm{C}x^{2n}+\mathrm{D}x^{3n}+$ &c. then will the ſum of the ſeries $\frac{\mathrm{A}}{\alpha\,.\,\beta\,.\,\gamma\,.\,\delta\,.\,\&c.}+\frac{\mathrm{B}x^{n}}{\alpha+n\,.\,\beta+n\,.\,\gamma+n\,.\,\&c.}+\frac{\mathrm{C}x^{2n}}{\alpha+2n\,.\,\beta+2n\,.\,\gamma+2n\,.\,\&c.}+\&c.=\frac{1}{\alpha\beta\gamma\delta,\,\&c.}\times\mathrm{P}-\frac{1}{\alpha}\,.\,\frac{1}{\beta-\alpha}\,.\,\frac{1}{\gamma-\alpha}\,.\,\frac{1}{\delta-\alpha}\,.\times\&c.\,x^{-\alpha}\int x^{\alpha}\dot{p}-\frac{1}{\beta}\,.\,\frac{1}{\alpha-\beta}\,.\,\frac{1}{\gamma-\beta}\,.\,\frac{1}{\delta-\beta}\,.\,\&c.\,x^{-\beta}\int x^{\beta}\dot{p}-\frac{1}{\gamma}\,.\,\frac{1}{\alpha-\gamma}\,.\,\frac{1}{\beta-\gamma}\,.\,\frac{1}{\delta-\gamma}\,.\,\&c.\times x^{-\gamma}\int x^{\gamma}\dot{p}-\&c.$

This may be proved from the ſubſequent arithmetical propoſition $\frac{1}{\alpha}\,.\,\frac{1}{\beta-\alpha}\,.\,\frac{1}{\gamma-\alpha}\,.\,\frac{1}{\delta-\alpha}\,.\,\&c.+\frac{1}{\beta}\,.\,\frac{1}{\alpha-\beta}\,.\,\frac{1}{\gamma-\beta}\,.\,\frac{1}{\delta-\beta}\,.\,\&c.+\frac{1}{\gamma}\,.\,\frac{1}{\alpha-\gamma}\,.\,\frac{1}{\beta-\gamma}\,.\,\frac{1}{\delta-\gamma}\,.\,\&c.+\frac{1}{\delta}\,.\,\frac{1}{\alpha-\delta}\,.\,\frac{1}{\beta-\delta}\,.\,\frac{1}{\gamma-\delta}\,.\,\&c.=\frac{1}{\alpha\,.\,\beta\,.\,\gamma\,.\,\delta\,.\,\&c.}$.

19. Let the general term of the above-mentioned ſeries $\mathrm{A}+\mathrm{B}x+\mathrm{C}x^{2n}+$&c. be $\mathrm{H}x^{zn}$; then from the ſums of the ſeries p, and the fluents of the fluxions $x^{\alpha}\dot{p}$, $x^{\beta}\dot{p}$, $x^{\gamma}\dot{p}$, $x^{\delta}\dot{p}$, &c. being given there follows the ſum of a ſeries, whoſe general term is $\frac{az^{ln}+bz^{ln-1}+cz^{ln-2}+\&c.}{\alpha+nz\,.\,\beta+nz\,.\,\gamma+nz\,.\,\delta+nz\,.\,\&c.}\times\mathrm{H}x^{zn}$, where l denotes a whole number.

If $\mathrm{H}=m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\ldots\frac{m-z}{z+1}$, or $=m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\ldots\frac{m-lz}{lz+1}$, where l is a whole number, and m, α', β', γ', &c. are either whole numbers or fractions whoſe denominator is 2, and $\alpha=\alpha'n$, $\beta=\beta'n$, &c. the ſum of the above-mentioned ſeries can be found by finite terms, circular arcs and logarithms.

If $H = \frac{1}{1 . 2 . 3 . . z}$ or $= \frac{1}{1 . 2 . 3 . . lz}$, and l, α, β, γ, &c. whole numbers and $n = 1$; then can the ſum of any ſeries of the above-mentioned formula be found in finite algebraical functions of x, and the circular arcs and logarithms of them.

PART II.

1. THE doctrine of proportional parts was probably very early known in the æra of ſcience; for when men could not find the exact value of a quantity, they were induced to find near approximations by trials, and from thence, by proportion, an approximation ſtill nearer: which method is commonly denominated the Rule of Falſe.

This was often found to deviate conſiderably from the exact value; and the ſame operation was repeated, which frequently produced a nearer approximate value, and ſo on.

This method of approximations, the moſt general yet known, has been uſed in reſolving problems by ſeveral of the moſt eminent mathematicians in different ages, and in this particularly by M. EULER.

2. The following obſervation, I believe, was firſt publiſhed in the *Meditationes*, in the year 1770, *viz.* that the convergency of the approximate values, found by the rule of falſe and method of infinite ſeries, generally depended on this, *viz.*

how much nearer the approximate aſſumed is to one value of the quantity ſought poſſible or impoſſible than to any other, and not to the quantity itſelf: hence, when two or more (n) values of the quantity ſought are nearly equal, it is neceſſary to recur to more difficult rules, *viz.* to three or more trials; as, for example, let two roots be nearly equal, and write a, $a+\pi$, and $a+\varrho$, for the unknown quantity in the given equation made $=0$, and let the quantities reſulting be A, B, and C, then will more near approximations to the two roots nearly equal of the given equation be $a+$ the two roots (x) of the quadratic $\left(\frac{A}{\pi\rho}+\frac{B}{\pi(\pi-\rho)}+\frac{C}{\rho(\rho-\pi)}\right)x^2-\left(A\times\frac{\pi+\rho}{\pi\rho}+B\times\frac{\rho}{\pi\,.\,(\pi-\rho)}+\frac{C\pi}{\rho\,.\,(\rho-\pi)}\right)x+A=0$: for write 0, π, and ρ, reſpectively for x in the equation, and there will reſult the quantities A, B, and C.

More generally, ſubſtitute for x in the given equation the quantities a, $a+\pi$, $a+\rho$, $a+\sigma$, $a+\tau$, $a+\upsilon$, &c. where π, ϱ, σ, τ, &c. are very ſmall quantities; and let the quantities reſulting be A, B, C, D, E, F, &c.; then will more near approximations to the (n) roots of the given equation be $a+\alpha$, $a+\beta$, $a+\gamma$, $a+\delta$, &c. where α, β, γ, δ, &c. &c. are the n roots (e) of the given equation

$$\frac{(e-\pi)(e-\varrho)(e-\sigma)(e-\tau)\&c.}{\pm\pi\varrho\sigma\tau,\ \&c.}\times A+\frac{e(e-\varrho)(e-\sigma)(e-\tau)\&c.}{\pi(\pi-\varrho)(\pi-\sigma)(\pi-\tau)\&c.}\times B+\frac{e(e-\pi)(e-\sigma)(e-\tau)\&c.}{\varrho(e-\pi)(\varrho-\sigma)(\varrho-\tau)\&c.}\times C+\&c.=A-\left(\frac{A}{\pi}-\frac{B}{\pi}\right)e+\left(\frac{A}{\pi\varrho}+\frac{B}{\pi(\pi-\varrho)}+\frac{C}{\varrho(\varrho-\pi)}\right)\times e\,.\,\overline{e-\pi}-\left(\frac{A}{\pi\varrho\sigma}-\frac{B}{\pi(\pi-\varrho)(\pi-\sigma)}-\frac{C}{\rho(\rho-\pi)(\rho-\sigma)}-\frac{D}{\sigma(\sigma-\pi)(\sigma-\varrho)}\right)\times e\,.\,\overline{e-\pi}\,.\,\overline{e-\rho}+\left(\frac{A}{\pi\rho\sigma\tau}+\frac{B}{\pi(\pi-\varrho)(\pi-\sigma)(\pi-\tau)}+\frac{C}{\varrho(\varrho-\pi)(\varrho-\sigma)(\varrho-\tau)\,\&c.}+\frac{D}{\sigma(\sigma-\pi)(\sigma-\varrho)(\sigma-\tau)\&c.}+\frac{E}{\tau(\tau-\pi)(\tau-\varrho)(\tau-\sigma)\&c.}\right)\,.\,e\,.\,\overline{e-\pi}\,.\,\overline{e-\varrho}\,.\,\overline{e-\sigma}-\left(\frac{A}{\pi\varrho\sigma\tau\upsilon}-\frac{B}{\pi(\pi-\varrho)(\pi-\sigma)(\pi-\tau)(\pi-\upsilon)}-\frac{C}{\varrho(\varrho-\pi)(\varrho-\sigma)(\varrho-\tau)}\right.$$

$$\frac{}{(\varrho-\tau)(\varrho-\upsilon)}-\frac{D}{\sigma(\sigma-\pi)(\sigma-\varrho)(\sigma-\tau)(\sigma-\upsilon)}-\frac{E}{\tau(\tau-\pi)(\tau-\varrho)(\tau-\sigma)(\tau-\upsilon)}-\frac{F}{\upsilon(\upsilon-\pi)(\upsilon-\varrho)(\upsilon-\sigma)(\upsilon-\tau)}\Big)e\,.\,\overline{e-\pi}\,.\,\overline{e-\rho}\,.\,\overline{e-\sigma}\,.\,\overline{e-\tau}+\&c.=0.$$

Their reſolutions were firſt given in the *Meditationes Analyticæ*, publiſhed in the year 1774, and require the extraction of a quadratic, cubic, and in general of an equation of (n) dimenſions; which rules will often give a nearer approximate than the preceding, when the roots are not nearly equal.

3. Theſe rules may be applied to find approximations to the roots of algebraical equations: for example, let the algebraical equation be $x^n-px^{n-1}+qx^{n-2}-\&c.=0$, ſubſtitute in it for x two quantities a, and $a+e$ much nearer to one root than to any other, and there reſult $a^n-pa^{n-1}+qa^{n-2}-\&c=A$, and $(a+e)^n-p(a+e)^{n-1}+q(a+e)^{n-2}-\&c.=B$; then, by the rule of falſe, $B-A:e::A:\dfrac{a^n-pa^{n-1}+qa^{n-2}-\&c.}{(na^{n-1}-\overline{n-1}pa^{n-2}+\overline{n-2}qa^{n-3}-\&c.)+\&c.}=b$; whence $a-b$ a near approximate value to the root ſought. If the quantities, in which are involved e, e^2, e^3, &c. on account of e being very ſmall, be rejected, then will the approximate ſought $b=$ $\dfrac{a^n-pa^{n-1}+qa^{n-2}-\&c.}{na^{n-1}-\overline{n-1}pa^{n-2}+\overline{n-2}qa^{n-3}-\&c.}$; which will nearly be the ſame as found, where a near approximate is given, from the method given by VIETA, HARRIOT, OUGHTRED, NEWTON, DE LAGNY, HALLEY, &c.

4. From this expreſſion it follows, that if (a) be a root of the equation $na^{n-1}-\overline{n-1}pa^{n-2}+\&c.=0$, of which the roots are limits between the roots of the given equation, the approximation found will be infinite.

5. In finding theſe approximations, when there are two or more quantities contained in the given equation dependent on

each other, as the arc and the ſine, it is neceſſary that both ſhould be corrected in every approximation to ſuch a degree as the ſubſequent approximations require.

6. In the *Meditationes* it is obſerved, that in any algebraical equation $x^n - ax^{n-1} + bx^{n-2} - cx^{n-3} + dx^{n-4} - ex^{n-5} \ldots \pm gx^{n-m+1} \mp hx^{n-m} \pm kx^{n-m-1} \mp lx^{n-m-2} \pm \&c. = 0$, if a be much greater than $\frac{b}{a}$, and $\frac{b}{a}$ has to $\frac{c}{b}$ a much greater ratio than $a : \frac{b}{a}$; and in the ſame manner $\frac{c}{b}$ has to $\frac{d}{c}$ a much greater ratio than $\frac{b}{a} : \frac{c}{b}$, and ſo on; then will a be a near approximate to the greateſt root of the algebraical equation; $\frac{b}{a}$ a near approximate to the ſecond; $\frac{c}{b}$ a near approximate to the third, and $\frac{k}{h}$ a near approximate to a root, which is much leſs than m roots of the given equation, but much greater than the remaining $(n-m-1)$ roots.

If the equation above-mentioned $x^n \pm ax^{n-1} + \&c. = 0$, or which is the ſame, $1 \pm \frac{a}{x} + \frac{b}{x^2} \pm \&c. = 0$ be infinite; then will, in like manner, all its roots be poſſible and their approximate values a, $\frac{b}{a}$, $\frac{c}{b}$, &c. as before.

This eaſily appears by ſubſtituting for a, b, c, &c. their values in terms of the root of the equation.

7. A nearer approximate to the above-mentioned root will be $\frac{k}{h} - \left(\frac{l}{k} - \frac{gk^2}{h^3}\right) + \&c.$

8. Equations, of which the fluxions of the quantities contained in the given equations can be found, may be reduced to infinite algebraical equations, in which is involved no irrational function of the unknown quantities contained in the given equations by the incremental theorem; *viz.* let $A = 0$ be the given

equation, and (a) an approximate much more near to the root required (π) of the given equation than to any other: write a for (x) the unknown quantity ſought in the ſubſequent quantities, A, $\frac{\dot{A}}{x}$, $\frac{\ddot{A}}{x^2}$, $\frac{\dddot{A}}{x^3}$, &c.; and let there reſult the correſpondent quantities α, β, γ, δ, &c.; then will $\pi - a$ be a root (e) of the infinite equation $\alpha \pm \beta e + \frac{1}{1 . 2} \gamma e^2 \pm \frac{1}{1 . 2 . 3} \delta e^3 + \frac{1}{1 . 2 . 3 . 4} \varepsilon e^4 \pm$ &c. $= 0$, of which a root of the equations $\alpha \pm \beta e = 0$, $\alpha \pm \beta e + \frac{1}{1 . 2} \gamma e^2 = 0$, &c. will be an approximate. If two roots of the given equation are nearly $= a$, then it is neceſſary to recur to an equation not inferior to a quadratic.

9. The ſucceſſive approximate values found by theſe and like rules will ultimately be to each other in a greater than any geometrical ratio: for example, let $\frac{as}{a+s}$ be an approximate to a root of the quadratic $x^2 - (a+s)x + as = 0$, then will the new addition to the approximate to the root s found by the common method at the diſtance $n - 1$ from the firſt approximate be $\frac{s^b}{a^{b-1}}$ nearly, where $b = 2^{n-1}$.

10. Let an equation $x^n - Px^{n-1} + Qx^{n-2} - Rx^{n-3} + Sx^{n-4} -$ &c. $= 0$, of which the roots are a, b, c, d, &c.; and an equation $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} -$ &c. $= 0$, where p, q, r, s, &c. differ from the co-efficients P, Q, R, S, &c. by very ſmall quantities: aſſume the (n) equations $\pi + \rho + \sigma + \tau +$ &c. $= p - P = \alpha$, $a\pi + b\varrho + c\sigma + d\tau +$ &c. $= P\alpha - q + Q = \beta$, $a^2\pi + b^2\varrho + c^2\sigma + d^2\tau +$ &c. $= P\beta - Q\alpha + r - R = \gamma$, $a^3\pi + b^3\rho + c^3\sigma + d^3\tau +$ &c. $= P\gamma - Q\beta + R\alpha - s + S = \delta$, &c.; and from them find the unknown quantities π, ϱ, σ, τ, &c; then will $a + \pi$, $b + \varrho$, $c + \sigma$, $d + \tau$, &c. be nearly the n roots of the equation $x^n - px^{n-1} + qx^{n-2}$

$-$&c. $=0$, and conſequently π, ϱ, &c. nearly $-\frac{a^n - pa^{n-1} + qa^{n-2} - \&c.}{na^{n-1} - \overline{n-1}pa^{n-2} + \&c.}$, $-\frac{b^n - pb^{n-1} + qb^{n-2} - \&c.}{nb^{n-1} - \overline{n-1}pb^{n-2} + \&c.}$, &c.; whence the convergency of the approximate values found by this rule depends on the principle before delivered.

10. Let there be given (m) equations, which contain m unknown quantities x, y, z, &c.; and let α, β, γ, &c. be nearly correſpondent values of the unknown quantities x, y, &c. reſpectively: aſſume $n+1$ different values of the quantity x, viz. α, $\alpha+\pi$, $\alpha+\pi'$, $\alpha+\pi''$, &c. &c.; and in like manner aſſume $n+1$ different correſpondent values of the quantity y, which let be β, $\beta+\varrho$, $\beta+\rho'$, $\beta+\rho''$, &c.; and ſo of the remaining; where π, π', π'', &c. ρ, ρ', ρ'', &c. &c. are very ſmall quantities; ſubſtitute theſe quantities for their reſpective values in the given equations, and let the reſulting quantities be A, B, C, D, &c. in the firſt equation; P, Q, R, S, &c. in the ſecond, &c. &c.: aſſume from the firſt equation the n ſimple equations $a\pi + b\varrho +$ &c. $= \mathrm{B} - \mathrm{A}$, $a\pi' + b\varrho' +$ &c. $= \mathrm{C} - \mathrm{A}$, $a\pi'' + b\varrho'' +$ &c. $= \mathrm{D} - \mathrm{A}$, &c.; and from the ſecond equation the n ſimple ones $h\pi + k\varrho +$ &c. $= \mathrm{Q} - \mathrm{P}$, $h\pi' + k\varrho' +$ &c. $= \mathrm{R} - \mathrm{P}$, $h\pi'' + k\varrho'' +$ &c. $= \mathrm{S} - \mathrm{P}$, &c. From theſe equations can be inveſtigated the co-efficients a, b, &c. h, k, &c. &c.; ultimately aſſume the m equations $\mathrm{A} + ae + bi +$ &c. $= 0$, $\mathrm{P} + he + ki +$ &c. $= 0$, &c. from which can be deduced the values of the quantities e, i, &c.; and $\alpha+e$, $\beta+i$, &c. will be more near values of the quantities, x, y, &c.

11. Sir ISAAC NEWTON found the ſum (A) of the $2n$th power of each of the roots of a given equation, and then extracted the $2n$th root of A, *viz.* $\sqrt[2n]{\mathrm{A}}$ for an approximate value of the greateſt root of the equation, and further added ſome ſimilar rules on the ſame principle.

In the *Miſcell. Analyt.* and *Meditationes* the ſame principle is applied in different rules for finding approximates to the greateſt and other roots of the given equation; and alſo limits of the ratios of the approximate values of the roots found by theſe rules to the roots themſelves are given.

It is obſerved in the *Meditationes*, that from theſe rules in general to find the greateſt root, it is often neceſſary that the greateſt poſſible root be greater than the ſum of the quantities contained in the poſſible and impoſſible part of any impoſſible root of the given equation: for example, if $a+b\sqrt{-1}$ be an impoſſible root of the given equation, then it is neceſſary that the greateſt poſſible root be greater than $a+b$.

It may further be obſerved, that in equations of high dimenſions (unleſs purpoſedly made) it is probable, the number of impoſſible will greatly exceed the number of poſſible roots; and conſequently theſe rules moſt commonly fail.

12. M. Bernoulli aſſumed a fraction whoſe numerator is a rational function of the unknown quantity, and denominator the quantity, which conſtitutes the equation; and reduced the fraction into a ſeries, whoſe terms proceed according to the dimenſions of the unknown quantity; and thence found an approximate value of the greateſt or leaſt root of the given equation or its reciprocal, by dividing the co-efficient of any term of the ſeries reſulting by the co-efficient of the preceding or ſubſequent term: for example, let the equation be $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} \,..\, \pm Px^3 \mp Qx^2 \pm Rx \mp S = 0$; aſſume the fraction $\frac{nx^{n-1} - \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-3} - \overline{n-3}\,rx^{n-4} + \&c.}{x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c.} =$ $nx^{-1} + (\alpha+\beta+\gamma+\delta+\&c.)\,x^{-2} + (\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2+\&c.)\,x^{-3} + (\alpha^3+\beta^3+\gamma^3+\delta^3+\varepsilon^3+\&c.)\,x^{-4} + (\alpha^4+\beta^4+\gamma^4+\delta^4+\varepsilon^4+\&c.)$

&c.)x^{-5}. . . . $+ (\alpha^{m-1} + \beta^{m-1} + \gamma^{m-1} + \&c. = \mathrm{P})x^{-m} + (\alpha^{m} + \beta^{m} + \gamma + \&c. = \mathrm{Q})\, x^{-m-1} + (\alpha^{m+1} + \beta^{m+1} + \gamma^{m+1} + \&c. = \mathrm{R})\, x^{-m-2} +$ &c.; then will $\frac{\mathrm{R}}{\mathrm{Q}}$ or $\sqrt[m]{\mathrm{Q}}$ be the greateſt root nearly.

Ex. 2. $\frac{\mathrm{R} - 2\mathrm{Q}x + 3\mathrm{P}x^{2} - \&c. \ldots nx^{n-1}}{\mathrm{S} - \mathrm{R}x + \mathrm{Q}x^{2} - \mathrm{P}x^{3} + \&c. \ldots x^{n}} = (\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \&c.) + (\frac{1}{\alpha^{2}} + \frac{1}{\beta^{2}} + \frac{1}{\gamma^{2}} + \frac{1}{\delta^{2}} + \&c.)\, x + (\frac{1}{\alpha^{3}} + \frac{1}{\beta^{3}} + \frac{1}{\gamma^{3}} + \&c.)\, x^{2} + (\frac{1}{\alpha^{4}} + \frac{1}{\beta^{4}} + \frac{1}{\gamma^{4}} + \&c.)\, x^{3} + \ldots\ldots + (\frac{1}{\alpha^{m-1}} + \frac{1}{\beta^{m-1}} + \frac{1}{\gamma^{m-1}} + \&c. = \mathrm{P})\, x^{m-2} + (\frac{1}{\alpha^{m}} + \frac{1}{\beta^{m}} + \frac{1}{\gamma^{m}} + \&c. = \mathrm{Q})\, x^{m-1} + (\frac{1}{\alpha^{m+1}} + \frac{1}{\beta^{m+1}} + \frac{1}{\gamma^{m+1}} + \&c.)\, x^{m} +$ &c.; then will $\frac{\mathrm{P}}{\mathrm{Q}}$ or $\sqrt[m]{\frac{1}{\mathrm{Q}}}$, the leaſt root nearly.

Ex. 3. $\frac{1}{x^{n} - px^{n-1} + qx^{n-2} - \&c.} = x^{-n} + (\alpha + \beta + \gamma + \delta + \&c.)x^{-n-1} + (\alpha^{2} + \beta^{2} + \gamma^{2} + \&c.(+\alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \&c.))\, x^{-n-2} + (\alpha^{3} + \beta^{3} + \gamma^{3} + \&c.(+\alpha^{2}\beta + \alpha^{2}\gamma + \beta^{2}\gamma + \gamma^{2}\alpha + \gamma^{2}\beta + \&c.) + \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta + \&c.))\, x^{-n-3} +$ &c.; and $\frac{\mathrm{S}}{\mathrm{S} - \mathrm{R}x + \mathrm{Q}x^{2} - \&c. \ldots x^{n}} = 1 + (\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \&c.)\, x + (\frac{1}{\alpha^{2}} + \frac{1}{\beta^{2}} + \frac{1}{\gamma^{2}} + \&c.(+\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} + \&c.))\, x^{2} +$ &c. in each of which all the numeral co-efficients are 1. The approximate values to the greateſt and leaſt root may be found in the ſame manner as before.

From the preceding examples it appears, that the ſame obſervations which have been applied to Sir ISAAC NEWTON'S method are equally applicable to M. BERNOULLI'S.

13. In the *Meditationes* this rule is further extended, *viz.* let the given equation involve irrational and other functions of the unknown quantity; reduce it ſo that no function of the unknown quantity (x) may be contained in the denominator, and let the reſulting equation be $\mathrm{A} = 0$. Aſſume a fraction

$\frac{\mathrm{B}}{\mathrm{A}}$,

$\frac{B}{A}$, whoſe numerator B is a finite rational and integral function of the unknown quantity; reduce $\frac{B}{A}$ into a ſeries proceeding according to the dimenſions of the unknown quantity: for example, let the ſeries be $A'x^{r} + B'x^{r+s} + Cx^{r+2s} + \ldots + Lx^{r+ls} + Mx^{r+\overline{l+1}s} + Nx^{r+\overline{l+2}s} + \&c.$; then *(exceptis excipiendis)* if s be negative, will the greateſt root be $\sqrt[s]{\frac{M}{L}}$ nearly; but, if s be affirmative, $\sqrt[s]{\frac{L}{M}}$ will be the leaſt root nearly. If l be infinite, then *(exceptis excipiendis*, as before-mentioned) the quantities $\sqrt[s]{\frac{M}{L}}$ and $\sqrt[s]{\frac{L}{M}}$ will be the above-mentioned roots accurately.

Theſe principles have been applied to find the remaining roots of the given equation as well as the greateſt and leaſt.

14. The rule of falſe has been found very uſeful in finding approximates to the two unknown quantities contained in two given equations, and has been applied to (n) equations having (n) different unknown quantities: for example, it has been obſerved, that if two or more (m) values of an unknown quantity (x) are nearly equal to each other and to its given approximate value (x'), the unknown quantity $v = x - x'$ will aſcend to two or more (m) dimenſions in one of the reſulting equations; or in more than one equations will be contained ſuch powers of the quantity (v), that if the more equations were reduced to one whoſe unknown quantity is v, the reſulting equation will contain (m) dimenſions of the quantity v. Hence it appears, that in this caſe alſo the convergency of the approximate values found will depend on the given approximate being much more near to one root than to any other.

15. When the given equations $A=0$, $B=0$, $C=0$, &c. contain irrational or other quantities whoſe fluxions can be found; and approximates (a, b, c, d, &c.) are given to each of the unknown quantities (x, y, z, v, &c.) contained in the given equations; let $a+x'=x$, $b+y'=y$, $c+z'=z$, $d+v'=v$, &c.: ſubſtitute in the quantities A, $\left(\frac{A^{\cdot}}{\dot{x}}\right)$, $\left(\frac{A^{\cdot}}{\dot{y}}\right)$, $\left(\frac{A^{\cdot}}{\dot{z}}\right)$, $\left(\frac{A^{\cdot}}{\dot{v}}\right)$, &c.; $\left(\frac{\ddot{A}}{\dot{x}^2}\right)$, $\left(\frac{\ddot{A}}{\dot{x}\dot{y}}\right)$, $\left(\frac{\ddot{A}}{\dot{x}\dot{z}}\right)$, $\left(\frac{\ddot{A}}{\dot{x}\dot{v}}\right)$, &c.; $\left(\frac{\dddot{A}}{\dot{x}\dot{y}\dot{z}}\right)$, $\left(\frac{\dddot{A}}{\dot{x}^2\dot{y}}\right)$, &c. for x, y, z, v, &c. reſpectively a, b, c, d, &c.; let the reſulting correſpondent values be A', $\left(\frac{{}'A^{\cdot}}{\dot{x}}\right)$, $\left(\frac{{}'A^{\cdot}}{\dot{y}}\right)$, &c. whence may be deduced the equation $A'+\left(\left(\frac{{}'A^{\cdot}}{\dot{x}}\right)x'+\left(\frac{{}'A^{\cdot}}{\dot{y}}\right)y'+\left(\frac{{}'A^{\cdot}}{\dot{z}}\right)z'+\left(\frac{{}'A^{\cdot}}{\dot{v}}\right)v'+\text{\&c.}\right)+\left(\frac{1}{2}\left(\frac{{}'A^{\cdot\cdot}}{\dot{x}^2}\right)x'^2+\frac{1}{2}\left(\frac{{}'A^{\cdot\cdot}}{\dot{y}^2}\right)y'^2+\frac{1}{2}\left(\frac{{}'A^{\cdot\cdot}}{\dot{z}^2}\right)z'^2+\text{\&c.}+\left(\frac{{}'A^{\cdot}}{\dot{x}\dot{y}}\right)x'y'+\left(\frac{{}'A^{\cdot\cdot}}{\dot{x}\dot{z}}\right)x'z'+\text{\&c.}\right)+\text{\&c.}=0$ in which no irrational, &c. function of the approximates x', y', z', &c. is contained; and in the ſame manner may the remaining equations $B=0$, $C=0$, &c. be transformed into others, in which no irrational function of the approximates (x', y', z', &c.) is contained, and from the reſulting equations may be found approximate values of the quantities x', y', z', &c.

If there be given only two equations $A=0$ and $B=0$ containing two unknown quanties x and y, and all the quantities of the reſulting equations, in which are contained more than one dimenſion of the quantities x' and y' be rejected, there will reſult the two equations $A'+\left(\frac{A'^{\cdot}}{\dot{x}}\right)x'+\left(\frac{A'^{\cdot}}{\dot{y}}\right)y'=0$ and $B'+\left(\frac{B'^{\cdot}}{\dot{x}}\right)x'+\left(\frac{B'}{\dot{y}}\right)y'=0$, from which may be found x' and y', the ſame as given by Mr. Simpson and others. When two or more values of the quantity x are nearly $=a$, then in a reſulting equation or equations, quantities of two or more dimenſions of the approximate x' are to be included.

PART III.

1. THE firſt algebraiſts divided quantities, and extracted their roots no further than the quantities themſelves: they did not perceive the utility of proceeding any further, otherwiſe the operation would have been the ſame continued. Mr. Gregory St. Vincent, and Mr. Mercator divided, and Sir Isaac Newton divided and extracted the roots of quantities (in which only one unknown quantity (x) is contained) by the operations then uſed by arithmeticians, into ſeries aſcending or deſcending, according to the dimenſions of x *in infinitum*. They clearly ſaw the utility of it in finding the fluents of fluxions, as Dr. Wallis and others ſome little time before had found the fluent of the fluxion $ax^{\frac{m}{n}}\dot{x}$; or, which is the ſame, the area of a curve whoſe ordinate is $ax^{\frac{m}{n}}$ and abſciſſa is x.

2. M. Leibnitz aſked from Mr. Newton the caſes in which the above-mentioned ſerieſes would converge; for it would be altogether uſeleſs when it diverges, and of little uſe when it converges ſlowly.

Caſ. 1. To this queſtion an anſwer, I believe, was firſt given in the *Meditationes*, viz. reduce the function to its loweſt terms; and alſo in ſuch a manner that the quantities contained in the numerator and denominator may have no denominator: make the denominator $Q=0$, and every diſtinct irrational quantity contained in it $=0$; and alſo every diſtinct irrational quantity H contained in the numerator $=0$; then, let α be the leaſt root affirmative or negative (but not $=0$) of the above-mentioned reſulting equations, the aſcending ſeries will always converge, if the value of x is contained between α and $-\alpha$; but if x be greater than α or $-\alpha$, the above-mentioned ſeries will not converge.

If the above-mentioned ſeries (S) be multiplied into $\dot{x}$, and its fluent found; then will the ſeries denoting the fluent contained between two values a and b of the quantity (x) converge, when a and b are both contained between α and $-\alpha$: the fluent always converges faſter than the ſeries S, the unknown quantity x having the ſame values in both.

Ex. Let $\int \frac{\dot{x}}{a+bx+cx^2+\ldots x^n} = Ax+\frac{1}{2}Bx^2+\frac{1}{3}Cx^3+$ &c. and the roots of the equation $a+bx+cx^2+\ldots x^n=0$ be α, β, γ, δ, &c.; then $\frac{\dot{x}}{a+bx+cx^2\ldots x^n}=\frac{\pi\dot{x}}{\alpha-x}+\frac{\varrho\dot{x}}{\beta-x}+\frac{\sigma\dot{x}}{\gamma-x}+\text{\&c.}=A\dot{x}+Bx\dot{x}+Cx^2\dot{x}+\text{\&c.}$; but $\frac{\pi}{\alpha-x}=\frac{\pi}{\alpha}+\frac{\pi x}{\alpha^2}+\frac{\pi x^2}{\alpha^3}+$ &c. *in infinitum*, $\frac{\varrho}{\beta-x}=\frac{\varrho}{\beta}+\frac{\varrho x}{\beta^2}+\frac{\varrho x^2}{\beta^3}+$ &c. &c.; the former ſeries converges when x is contained between α and $-\alpha$, the latter when x is between β and β, and ſo on. In the ſame manner the fluents $\int\frac{\pi\dot{x}}{\alpha-x}=\frac{\pi}{\alpha}x+\frac{\pi x^2}{2\alpha^2}+$ &c. $\int\frac{\varrho\dot{x}}{\beta-x}=\frac{\varrho x}{\beta}+\frac{\varrho x^2}{2\beta^2}+$ &c. &c. *a fortiori* converge when x is between α and $-\alpha$, β and $-\beta$, &c. reſpectively, and ſo on: hence the ſeries $Ax+\frac{1}{2}Bx^2+\frac{1}{3}Cx^3+$ &c.

$=\int\frac{\dot{x}}{a+bx+cx^2\,..\,x^n}=\int\frac{\pi\dot{x}}{\alpha-x}+\int\frac{\varrho\dot{x}}{\beta-x}+\int\frac{\sigma\dot{x}}{\gamma-x}+\text{\&c.}=\left(\frac{\pi}{\alpha}+\frac{\varrho}{\beta}+\frac{\sigma}{\gamma}+\text{\&c.}\right)x+\left(\frac{\pi}{\alpha^2}+\frac{\varrho}{\beta^2}+\frac{\sigma}{\gamma^2}+\text{\&c.}\right)x^2+\text{\&c.}$ always converges when x is between α and $-\alpha$, where α is the leaſt root of the above-mentioned equation; but where x is greater than α or $-\alpha$, the ſeries will diverge.

The infinite ſeries $a^m+ma^{m-1}x+m\,.\,\frac{m-1}{2}\,a^{m-2}x^2+\text{\&c.}=\overline{a+x}^m$ will always converge when a is greater than x, and diverge when leſs, and conſequently its convergency does not depend on the index m, unleſs when $x=\pm a$: and in the *Meditationes Analyticæ* are given the caſes in which it converges or diverges when $\mp a=x$; and alſo if the ſeries $x^m+max^{m-1}+\text{\&c.}=\overline{x+a}^m$ deſcends according to the dimenſions of x, when it converges or diverges.

Caſ. 2. Let the above-mentioned quantity be reduced into a ſeries $Ax^{-r}+Bx^{-r-1}+\text{\&c.}$ deſcending according to the dimenſions of the unknown quantity x; then will the ſeries $Ax^{-r}+Bx^{-r-1}+\text{\&c.}=P$, or the ſeries $\frac{Ax^{-r+1}}{1-r}-\frac{Bx^{-r}}{r}+\text{\&c.}=\int P\dot{x}$ converge, when x is greater than the greateſt root (λ) of the above-mentioned equations, and diverge when it is leſs; and conſequently in this caſe, when the fluent is required between the two values a and b of x; the ſeries found will converge when a and b are both greater than λ.

Caſ. 3. When x is equal to the leaſt root in the former caſe, and to the greateſt in the latter, then ſometimes the ſeries will converge, and ſometimes not. Theſe different caſes are given in the *Meditationes*; but it would be too long to inſert them in this Paper.

4. If any roots are impoſſible as $a-b\sqrt{-1}$ and $a+b\sqrt{-1}$, then the ſeries will converge when x is in the firſt caſe

leſs than $\pm(a-b)$, and in the ſecond caſe greater than $\pm(a+b)$; or, more general, it will converge in the firſt caſe when x^n is always infinitely leſs than the reciprocal of the quantity $\frac{(a+b\sqrt{-1})^n+(a-b\sqrt{-1})^n=P}{(a^2+b^2)^n}$, where n is infinite; and in the latter caſe it will converge when x^n is infinitely greater than P.

It may not be improper to obſerve, that the ſame values of the root are to be applied in the equations, which are applied in the ſeries.

3. Sir ISAAC NEWTON, in the binomial theorem, reduced the power or root of a binomial into a ſeries proceeding according to the dimenſions of the terms contained in the binomial. M. DE MOIVRE reduced the power or root of a multinomial into a like ſeries; but in all caſes, except the moſt ſimple, we muſt ſtill recur to the common diviſion, extraction of roots, &c.

4. Meſſ. EULER, MACLAURIN, and other mathematicians, finding that the ſerieſes before-mentioned often converged ſlowly, or, if the truth may be confeſſed, commonly not at all, to deduce the area of a curve contained between two values a and b of the abſciſs, or fluent of a fluxion between two values a and b of the variable quantity x, interpolated the ſeries or area between a and b; that is, found the area or fluent contained between the abſciſſæ a and $a+\alpha$, then between the abſciſſæ $a+\alpha$ and $a+2\alpha$, and then between the abſciſſæ $a+2\alpha$ and $a+3\alpha$, and ſo on, till they came to the area between $b-\alpha$ and b. M. EULER obſerved, that when the ordinate became 0 or infinite, the ſeries expreſſing the area converges ſlowly; and therefore,

in order to inveſtigate the area near the points of the abſciſs, where the ordinates become o or infinite, he transforms the equation, and finds ſerieſes expreſſing the area near thoſe points, in which ſerieſes the abſciſſæ or unknown quantities begin from thoſe points.

5. In the *Meditationes* it is aſſerted, that in a ſeries proceeding according to the dimenſions of x, if any root of the above-mentioned equations be ſituated between the beginning of the abſciſs o and its end x, the ſeries will not converge: it is therefore neceſſary to transform the abſciſs ſo that it may begin or end at each of the roots of the above-mentioned equations, and conſequently where the ordinates become o or infinite, &c.; thoſe caſes excepted where the ordinate becomes o, and its correſpondent abſciſſa is a root of a rational function W of x without a denominator, and $\int W P\dot{x}$ is equal to the given ſeries; and in general the abſciſſæ ought to begin from the above-mentioned points; for if they end there the ſeries will converge very ſlow, if at all.

6. It is further aſſerted, that if a and b, the values of the abſciſſæ between which the area is required, be both more near to one root of the above-mentioned equations than to any other, and n ſerieſes are to be found, whoſe ſum expreſs the area contained between a and b; then that the ſum of the (n) ſerieſes may converge the ſwifteſt, the diſtances of the beginnings of each of the n abſciſſæ from the adjacent root will form a geometrical progreſſion.

7. Mr. CRAIG found the fluent of any fluxion of the formula $(a+bx^n+cx^{2n}+\&c.)^m x^{\theta-1}\dot{x}$ by a ſeries of the following

kind $(a+bx^{n}+cx^{2n}+\&c.)^{m+1}\times x^{\theta}\times(\alpha+\beta x^{n}+\gamma x^{2n}+\&c.$ *in infinitum)*. Sir ISAAC NEWTON, by ſerieſes of the ſame kind, found the fluents of fluxions of this formula $(a+bx^{n}+cx^{2n}+\&c.)^{l}\times(e+fx^{n}+gx^{2n}+\&c.)^{m}\times\&c.\ x^{\theta-1}\dot{x}$; the ſame principle is extended ſomewhat more general in the *Meditationes*.

8. Mr. JOHN BERNOULLI found the fluent of any fluxion $\int n\dot{z}=nz-\frac{z^{2}\dot{n}}{2\dot{z}}+\frac{z^{3}\dot{\dot{n}}}{2\,.\,3\dot{z}^{2}}-\&c.$ from the principles which Mr. CRAIG publiſhed for finding the fluents of fluxions involving fluents.

In the *Meditationes* ſomething is added of the convergency of theſe ſeries; and alſo,

9. In them a new method is given of finding approximations. Let ſome terms in the given quantity be much leſs or greater than the reſt; then reduce the quantity into terms proceeding according to the dimenſions of the ſmall quantities, or according to the reciprocals of the great quantities, and it is done. If the fluent of the quantity reſulting is required, find it from the common methods, if poſſible; but if not, reduce the terms not to be found into an infinite ſeries, and then find approximate values to each of the terms, &c.

Ex. 1. Let R the radius, and A the arc of a circle whoſe ſine is S and coſine C, and A$\pm e$ an arc of a circle which does not much differ from the arc A, that is, let e be a very ſmall quantity: then will the ſine of the arc A$\pm e$ be $\mathrm{S}\left(1-\frac{1}{2}\times\frac{e^{2}}{\mathrm{R}^{2}}+\frac{1}{2\cdot3\cdot4}\times\frac{e^{4}}{\mathrm{R}^{4}}-\frac{1}{2\cdot3\cdot4\cdot5\cdot6}\times\frac{e^{6}}{\mathrm{R}^{6}}+\&c.\right)\pm\mathrm{C}\left(\frac{e}{\mathrm{R}}-\frac{1}{2\cdot3}\times\frac{e^{3}}{\mathrm{R}^{3}}+\frac{1}{2\cdot3\cdot4\cdot5}\,\frac{e^{5}}{\mathrm{R}^{5}}-\&c.\right)$; and the coſine of the ſame arc will be $\mathrm{C}\left(1-\frac{1}{2}\times\frac{e^{2}}{\mathrm{R}^{2}}+\frac{1}{2\cdot3\cdot4}\times\frac{e^{4}}{\mathrm{R}^{4}}-\&c.\right.$ *in infinitum*$\left.\right)\pm\mathrm{S}\left(\frac{e}{\mathrm{R}}-\right.$

$\frac{1}{2 \cdot 3} \times \frac{e^3}{R^3} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} \times \frac{e^5}{R^5} - \&c.).$

Ex. 2. Log. $(x \pm e) = \int \frac{\dot{x}}{x \pm e} = \text{log.}\, x \pm \frac{e}{x} \mp \frac{e^2}{2x^2} \pm \frac{e^3}{3x^3} \mp \&c.:$
$\int \frac{\dot{x}}{a^2 - (x+e)^2} = \int \frac{\dot{x}}{a^2 - x^2} + \frac{e}{a^2 - x^2} + \&c.$

Ex. 3. $\int \frac{\dot{x}}{a^2 + (x-e)^2} = \int \frac{\dot{x}}{a^2 + x^2} - \frac{e}{a^2 + x^2} + \&c.$

Ex. 4. $\int \frac{\dot{y}}{\sqrt{(1-(y \pm e)^2)}} = \int \frac{\dot{y}}{\sqrt{1-y^2}} \pm \frac{e}{\sqrt{1-y^2}} \pm \&c.$

Ex. 5. $\int \frac{\dot{y}}{\sqrt{1-y^2} + e} = \int \frac{\dot{y}}{\sqrt{1-y^2}} - \int \frac{e\dot{y}}{1-y^2} + \int \frac{e^2\dot{y}}{(1-y^2)^{1\frac{1}{2}}} -$
$\int \frac{e^3\dot{y}}{(1-y^2)^2} + \&c.$

Ex. 6. Let the fluxion be $\frac{\sqrt{1-cx^2}\,\dot{x}}{\sqrt{1-x^2}}$, where c is a very ſmall quantity; then, if P be put for $\sqrt{1-x^2}$, the fluxion becomes $\frac{\dot{x}}{P} - \frac{cx^2\dot{x}}{2P} - \frac{c^2x^4\dot{x}}{8P} - \frac{c^3x^6\dot{x}}{16P} - \&c.$ of which the fluents will be found $A - \frac{c}{2} \times \frac{1 \cdot A - xP}{2} - \frac{c^2}{8} \times \frac{3B \quad x^3P}{4} - \frac{c^3}{16} \times \frac{5C - x^5P}{6} - \&c.$ where $A = \int \frac{\dot{x}}{P}$, $B = \frac{1 \cdot A - xP}{2}$, $C = \frac{3B - x^3P}{4}$, &c.

This is the ſwifteſt converging ſeries for finding the length of the arc of an ellipſe nearly circular, which is yet known; for example, let the abſciſs to the axis beginning from the center $= x$, the ſemi-tranſverſe axis of the ellipſe be 1, its ſemi-conjugate $1 - d$; then will $c = 2d - d^2$, and let the length of the quadrant of the ellipſe be required, in this caſe $x = 1$, and $P = \sqrt{1-x^2} = 0$; and $A = \frac{3,14159, \&c.}{2} = 1,57079$, &c. whence the length required is $1,57079, \&c. \times (1 - \frac{c}{2 \cdot 2} - \frac{1 \cdot 3c^2}{2^2 \cdot 4^2} - \frac{1 \cdot 3^2 \cdot 5c^3}{2^2 \cdot 4^2 \cdot 6^2} - \frac{1 \cdot 3^2 \cdot 5^2 \cdot 7c^4}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \&c.).$

Ex. 7. Let the fluxion be $\dot{x}(a^3+a'+(3a^2b+b')x+(3ab^2+c')x^2+(b^3+d')x^3)^m=\mathrm{P}\dot{x}$, where a', b', c', &c. are very ſmall quantities; then will $\mathrm{P}\dot{x}=((a+bx)^{3m}+m\overline{a+bx}^{3m-3}\times(a'+b'x+c'x^2+d'x^3)+m\,.\,\frac{m-1}{2}\,\overline{a+bx}^{3m-6}\times(a'+b'x+c'x^2+d'x^3)^2+\&c.)\,\dot{x}$, of which the fluent is $\frac{1}{3m+1\,.\,b}\,\overline{a+bx}^{3m+1}+m\,(a+bx)^{3m-2}\times\left(\frac{d'}{(3n+1)b}x^3+\frac{c'-3a\mathrm{A}}{3mb}x^2+\frac{b'-2a\mathrm{B}}{(3m-1)b}x+\frac{a'-a\mathrm{C}}{(3m-2)b}\right)+\&c.$ where the letters A, B, C, &c. denote the preceding co-efficients.

10. M. **EULER** and others, reduced the ſeries $\mathrm{A}x^r+\mathrm{B}x^{r+s}+\mathrm{C}x^{r+2s}+\&c.$ into a ſeries A′ ſin. $r\alpha$ + B ſin. $\overline{r+s}\,\alpha$ + &c. &c. where α denotes the arc of a circle, whoſe ſine is ax, &c. It may be eaſily reduced into infinite other ſerieſes proceeding according to the dimenſions of quantities, which are functions of x; but it is moſt commonly preferable to reduce it into ſerieſes proceeding according to the ſines, coſines, tangents, or ſecants of the arcs of circle, which ſines, &c. can immediately be procured from the common tables.

It has been obſerved in the firſt part, that to find the root of an equation, an approximate value much more near to one root of the equation than to any other muſt be given. In this part it is further obſerved, that ſerieſes deduced from expanding given quantities, ſo as to proceed according to the dimenſions of the unknown or variable quantities, will not converge if the unknown quantities be greater than the leaſt roots of the abovementioned equations; and that they will not converge much, unleſs the unknown quantities have a ſmall proportion to the leaſt roots: and if the given quantities be expanded into ſerieſes deſcending according to the dimenſions of the unknown quantities, then the ſerieſes reſulting will not converge if the greateſt roots of the equations before-mentioned be greater than the unknown quantities; and unleſs the unknown quantities have a great ratio to the greateſt roots the ſerieſes will converge ſlowly: for example, the ſerieſes $\int \frac{\dot{x}}{1+x} = x - \frac{1}{2}x^2 + \&c.$, $\int \frac{\dot{z}}{1+z^2} = z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \&c.$, $\int \frac{\dot{y}}{\sqrt{1-y^2}} = y + \frac{1}{6}y^3 + \&c.$ will never converge if x, z, or y, be greater than 1; but will always converge when leſs than ± 1 or $\pm 1\sqrt{-1}$ the leaſt or only roots of the equations $1+x=0$, $1-y^2=0$, and $1+z^2=0$. The ſeries $y+\frac{1}{6}y^3+\&c.$ will always converge when y is ſituated between $+1$ and -1, in which caſe alone it is poſſible. The ſame is true alſo of a ſeries ariſing from expanding the $\int (ax^m + bx^{m-1} + cx^{m-2} + \&c.)^{\frac{1}{2}m}\dot{x}$ into a ſeries proceeding according to the dimenſions of x, if the equation $ax^m + bx^{m-1} + cx^{m-2} + \&c. = 0$ has only two poſſible roots α and $-\alpha$, which are leſs in the manner before-mentioned than any impoſſible root contained in it.

If in either of the above-mentioned ſerieſes the unknown quantity x, z, or y, has a great proportion to 1, the ſeries will converge very ſlow; for example, if $x=1$, ten thouſand numbers at leaſt are to be calculated, to procure the ſum of the ſeries true to four figures; therefore, in theſe and moſt other ſerieſes it is neceſſary firſt to find a near value, *viz.* when x either $=z$, when e is very ſmall; or $=e$, when z is very ſmall; and then write $z+e$ for x in the quantity, and reduce it in the former caſe into a ſeries proceeding according to the dimenſions of e, in the latter caſe according to the dimenſions of z, and there will ariſe two ſerieſes, of which the fluents properly corrected, *viz.* by adding the fluent contained between the values a and e to the latter, and that between a and z to the former, will give the ſame fluent. The firſt term of the ſeries, in which e is ſuppoſed very ſmall, will be the fluent of the given fluxion, when $x=z$.

11. If a fluxion $P\dot{x}$, where P is a function of x, be transformed into another $Q\dot{z}$, where Q is a function of z, and they be reduced into ſerieſes A and B, proceeding according to the dimenſions of x and z reſpectively; find α and π, correſpondent values of the quantities x and z; then in aſcending ſerieſes, if α has a leſs ratio to the leaſt root of the equation $P=0$, than π has to the leaſt root of the equation $Q=0$, the ſeries A *(exceptis excipiendis)* will converge ſwifter than the ſeries B.

12. Dr. BARROW, in ſome equations, expreſſing the relation between the abſciſs x and ordinate y, found y in the two firſt terms of x, viz. $y=a+bx$, which is an equation to the aſymptotes of the curves. Sir ISAAC NEWTON, from an algebraical equation given, expreſſing the relation between y and x, found a ſeries proceeding according to the dimenſions of x, expreſſing y in terms of x. M. LEIBNITZ performed the ſame problem by aſſuming a ſeries $Ax^{n}+Bx^{n+r}+Cx^{n+2r}+$&c. with general co-efficients, and ſubſtituting this ſeries for y in the given equation, &c. from equating the correſpondent terms he deduced the indexes and co-efficients. M. DE MOIVRE, Mr. MAC LAURIN, &c. obſerved, that when the higheſt terms of the given equations have two or more (m) diviſors equal; for example, $(y-ax^{n})^{m}$; to which we muſt add, and when a value of y in this caſe is required nearly equal to Ax^{n}, a ſeries $Ax^{n}+Bx^{n+\frac{r}{m}}+$ &c. is to be aſſumed, whoſe indexes differ only by $\frac{r}{m}$, &c. if otherwiſe they would differ by r.

This reduction ſeldom anſwers any other purpoſe than finding the fluents of fluxions as $\int y\dot{x}$, &c.; or the aſymptotes, &c.

of curves, which depend on ſome of the firſt terms of the ſeries; but will very ſeldom be uſed for finding the roots of an equation; the rule of falſe, or method given by VIETA, will ever be ſubſtituted in its ſtead.

13. The values of x may be required between which the above-mentioned ſeries $Ax^n + Bx^{n+r} + Cx^{n+2r} +$ &c. will converge, as the infinite ſeries anſwers no purpoſe when it diverges. Firſt, if an aſcending be required, write for y in the given algebraical equation an infinite quantity, and find the roots of x in the equation thence reſulting $P = 0$; which for y write in the ſame equation, and find the roots of x in the reſulting equation which contain irrational quantities, *viz.* if one root be $x = a$; then let it contain $(x-a)^m$, where m is not a whole number; find the roots of the equations reſulting from making every irrational function of (x) contained in the given equation $= 0$, there being no irrational function of y contained in it; then, if x be greater than the leaſt root not $= 0$ of the above-mentioned equations, the ſeries will not converge; but if it be a ſeries deſcending according to the dimenſions of x, and x be leſs than the greateſt root of the above-mentioned equations, the ſeries will not converge.

In interpolating ſerieſes to inveſtigate the fluent contained between two values a and b of the fluxion $(Ax^n + Bx^{n+r} +$ &c.$)\dot{t}\dot{x}$, it is preferable to make the abſciſſæ begin from every one of the above-mentioned roots contained between a and b.

Moſt commonly theſe ſerieſes will not converge unleſs x be leſs, &c. than other quantities not inveſtigated by this rule.

14. Sir ISAAC NEWTON gave an elegant example of this rule in the reverſion of the ſeries, $y = ax + bx^2 + cx^3 +$ &c. from which the inveſtigation of the law of the ſeries has never been attempted. In the year 1757 I ſent the firſt edition of my

Meditationes

Meditationes Algebraicæ to the Royal Society, and publiſhed it in 1760, and afterwards in 1762, with another part added, on the Properties of curves, under the title of *Miſcellanea Analytica*, in all which was given the law of a ſeries for finding the ſum of the powers of the roots of an equation from its co-efficients. That great mathematician M. LE GRANGE and myſelf printed about the ſame time an obſervation known to me at the time that I printed the above-mentioned book, that the law of its powers and roots, if it proceeds *in infinitum*, is the ſame; from which ſeries of mine, with great ſagacity, M. LE GRANGE found the law which Sir ISAAC NEWTON's reverſion of ſeries obſerves. In the *Meditationes* the law is given, and the ſeries is made to proceed according to the dimenſions of x, &c.

15. It is aſſerted in the *Meditationes*, that in moſt equations of high dimenſions, unleſs purpoſedly conſtituted, the ſum of the terms which, from the given hypotheſis, become the greateſt, being ſuppoſed $=0$, only an approximate to the value Ax^n of y in the reſulting equation can by the common algebra be deduced. In this caſe the approximate to the quantity A is to be found ſo near as the approximate value of the quantity ſought requires; or perhaps it is preferable to correct in every operation the approximate values of the quantities A, B, C, &c. in the ſeries required $A'x^n + B'x^{n+r} + C'x^{n+2r} + \&c.$

In the equation the quantity $z \pm e$ may be ſubſtituted for x, and from the equation reſulting a ſeries expreſſing the value of y may be found, proceeding either according to the dimenſions of the quantity z, or its reciprocals, according to the conditions of the problem.

16. If there are more than one (n) equations having ($n+1$) unknown quantities (x, y, z, &c.): in each of the equations for y,

z, &c. write reſpectively Ax^n, $A'x^m$, &c.; and ſuppoſe the terms of each of the equations, which reſult the greateſt from the given or aſſumed hypotheſis $=0$, and from the reſulting equations may be found the firſt approximates Ax^n, $A'x^m$, &c. either accurately or nearly; then, in the given equations for y, z, &c. write $y' + (A + a)\, x^n + Bx^{n+n'} + \&c. z' + (A' + a')\, x^m + B'x^{m+m'}$, where a, a', &c. are very ſmall quantities; and ſuppoſe the terms of each of the equations which become greateſt from the above-mentioned hypotheſis reſpectively $=0$, and from the equations reſulting deduce the quantities a, a', &c.; n', m', &c.; B, B′, &c.; and ſo on: or aſſume $y = (A + 1a + a1 + \&c.)\, x^n + (B + 1b + b1 + \&c.)\, x^{n+n1} + \&c.$; $z = (A' + 1a' + a'1 + \&c.) x^m + (B' + 1b' + b'1 + \&c.) x^{m+m1} + \&c.$ &c.; ſubſtitute theſe quantities for their values in the given equations, and from equating the correſpondent terms of the reſulting equations may be deduced the quantities required.

The differences of the indexes n', &c. m', &c. may be deduced by writing x^n, x^m, &c. for y, z, &c. in the given equations, from the differences of the indexes of the quantities reſulting. The ſame principles may be applied in finding the above-mentioned differences, when two or more values are Ax^n, &c. which were applied in a like caſe to one equation having two unknown quantities.

The ſame principles which diſcover the caſes in which a ſeries deduced from an equation having two unknown quantities will converge, may be applied for the ſame purpoſe to theſe ſeries.

17. In the equations for x, y, z, &c. write reſpectively $x' + e$, $y' + f$, $z' + g$, &c.; and from the equations reſulting find y', z', &c. in ſerieſes either proceeding according to the dimenſions

of the quantities e, f, g, &c.; or according to the dimenſions of the quantity x', as the conditions of the problem require.

18. Given one or more (n) algebraical equations involving ($n+m$) unknown quantities, one unknown quantity (y) may be expreſſed by a ſeries proceeding according to the dimenſions of the $m-1$ remaining ones (x, z, v, &c.), in which any dimenſions of z, v, &c. aſſumed may be ſuppoſed to correſpond to (l) dimenſions of the quantity (x).

19. In a fluxional equation of the m^{th} order, expreſſing the relation between x, y, and their fluxions, where $\dot{x}$ is conſtant, Mr. EULER ſubſtitutes in the given equation for $\dot{y}^{m}$, $\dot{y}^{m-1}$, $\dot{y}^{m-2}$, $\dot{y}^{m-3}$, &c. reſpectively $Ax^{n-m}\dot{x}^{m}$, $\frac{A}{n-m+1}x^{n-m+1}\dot{x}^{m-1}+a\dot{x}^{m-1}$, $\frac{A}{(n-m+1)(n-m+2)}x^{n-m+2}\dot{x}^{m-2}+ax\dot{x}^{m-2}+b\dot{x}^{m-2}$, $\frac{A}{(n-m+1)(n-m+2)}$ $\frac{}{(n-m+3)}x^{n-m+3}\dot{x}^{m-3}+\frac{1}{2}ax^{2}\dot{x}^{m-3}+bx\dot{x}^{m-3}+c\dot{x}^{m-3}$, &c. where a, b, c, &c. are any quantities to be aſſumed in ſuch a manner as the conditions of the problem require; from ſuppoſing the aggregate of the terms of the reſulting equation, which are the greateſt, $=0$, may be deduced the firſt approximate Ax^{n}, or elſe (as is beforementioned) $A'x^{n}$ a near approximate to Ax^{n}, and by proceeding as in algebraical equations another approximate may be found, and ſo on. The ſame may be found by aſſuming $y=Ax^{n}+Bx^{n+r}+Cx^{n+2r}+\&c.+ax^{m}+bx^{m-1}+cx^{m-2}+\&c.$ or $y=(A+1a+a1+\&c.)x^{n}+(B+1b+b1+\&c.)x^{n+r}+(C+1c+c1+\&c.)x^{n+2r}+\&c.+ax^{m}+bx^{n-1}+cx^{m-2}+\&c.$ and ſubſtituting it and its fluxions for their values y, $\dot{y}$, $\ddot{y}$, &c. in the given equation, and ſuppoſing the aggregate of each correſpondent terms, which do not very much differ from each other, $=0$; from the reſulting equations can be deduced the co-efficients A, B, C, &c.; or A, $1a$, $a1$, &c.; B, $1b$, $b1$, &c.; C, $1c$, $c1$, &c. &c.

In the given equation for y, x, and their fluxions ſubſtitute $y'+f$, $x'+g$, and their fluxions, where the quantities f and g, &c. are aſſumed, ſo as to render the quantities y' and x', &c. very ſmall.

In finding the ſeries which expreſſes the value of y in terms of x, there will always occur as many invariable quantities to be aſſumed at will as is the order of the fluxional equation, provided the ſeries begins from its firſt terms; and to find them there will reſult equations eaſily reducible to homogeneous fluxional equations, of which the orders do not exceed m.